On New Laplacian Matrix with a User-Assigned Nullspace in Distributed Control of Multiagent Systems

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***Abstract*— Most distributed control results utilize the benchmark consensus algorithm, which is built on the well-known Laplacian matrix whose nullspace spans the vector of ones. Since this algorithm is the key building block for many distributed control architectures, extensions of this algorithms are also predicated on this Laplacian matrix. To this end, we explore how one can generalize the Laplacian nullspace, which can span any vector with positive elements, to pave the way for composing complex cooperative behaviors in multiagent systems. Specifically, a new Laplacian matrix is introduced for undirected and connected graphs that generalizes the well-known, standard Laplacian matrix, where it is based on a desired, user-assigned nullspace. We first give the mathematical definition of this Laplacian matrix and show that it inherits some fundamental properties of the standard Laplacian matrix. We then present distributed control architectures for convergence to the desired nullspace and for convergence to a specific vector within that nullspace. Finally, an application of the proposed Laplacian matrix to formation tracking and scaling problem is given.**

I. INTRODUCTION

Various applications of multiagent systems in civilian and military domains such as surveillance, reconnaissance, ground and air traffic management, payload and passenger transportation, task assignment, rapid internet delivery, and emergency response, to name but a few examples (e.g., see [1]–[5]), are developed based on the distributed control architectures that are built on the well-known Laplacian matrix whose nullspace spans the vector of ones (e.g., see [6]–[18]). To elucidate this point, consider the consensus algorithm over undirected and connected graphs with scalar integrator dynamics given by $\dot{x}_i(t) = -\sum_{i\sim j}\big(x_i(t) - x_j(t)\big)$, where $x_i(t)$ denotes the state of agent i, $i = 1,\ldots,N$, and $i \sim j$ indicates that agents i and j are neighbors. Defining $x(t) \triangleq [x_1(t),\ldots,x_n(t)]^{\mathrm{T}}$, one can compactly write the overall dynamics as $\dot{x}(t) = -\mathcal{L}x(t)$, where $\mathcal{L} \triangleq \mathcal{D} - \mathcal{A}$ is the Laplacian matrix with $\mathcal{D} \in \mathbb{R}^{n\times n}$ standing for its degree matrix and $\mathcal{A} \in \mathbb{R}^{n\times n}$ standing for its adjacency matrix (we also refer to the first paragraph of Section II for details on notation). In particular, the spectrum of the corresponding Laplacian matrix can be ordered as $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \le \ldots \le \lambda_n(\mathcal{L})$ ($\lambda_2(\mathcal{L})$ is the Fiedler eigenvalue determining the convergence rate), the null-space of this Laplacian matrix spans $\mathbf{1}_n = [1,\ldots,1]^{\mathrm{T}}$ ($\mathbf{1}_n$ is the eigenvector corresponding the zero eigenvalue $\lambda_1(\mathcal{L})$), and $\lim_{t\to\infty} x(t) = c\mathbf{1}_n$ with c being a scalar (the consensus value). Note that the above consensus algorithm is the key building block for many distributed control architectures including but not limited to formation architectures, pinning architectures, containment architectures, and dynamic information fusion architectures. As a consequence, all these extensions are also predicated on this Laplacian matrix with a nullspace spanning the vector of ones. The following question is now immediate: *To pave the way for composing complex cooperative behaviors in multiagent systems, can we generalize the Laplacian nullspace such that it can span any vector with positive elements?*

In this paper, we address the above question. Specifically, where we introduce a new Laplacian matrix for undirected and connected graphs that generalizes the well-known Laplacian matrix (hereinafter referred to as the standard Laplacian matrix) whose nullspace spans the vector of ones. The proposed Laplacian matrix is based on a desired, user-assigned nullspace. The mathematical definition of this Laplacian matrix is given and shown that it inherits some fundamental properties of the standard Laplacian matrix. Next, distributed control architectures for convergence to the desired nullspace and for convergence to a specific vector within that nullspace are presented. Finally, an application of the proposed Laplacian matrix to formation tracking and scaling problem is given.

Note that the authors of [19], [20] also investigate how to drive a given multiagent system to different Laplacian nullspace for undirected and connected graphs. They utilize a similarity transformation onto the standard Laplacian matrix to change its resulting nullspace, where this process leads to the same, standard degree matrix but to a new adjacency matrix. In contrast, the approach of this paper is based on keeping the same, standard adjacency matrix and altering the degree matrix instead. In other words, considering a distributed control architecture developed based on the standard Laplacian matrix, one can simply add self-loops to that architecture to achieve convergence to a given user-assigned nullspace based on the results of this paper; however, the results in [19], [20] require the exact knowledge of each neighboring agent states for the same purpose.

II. NEW LAPLACIAN MATRIX AND THE NULLSPACE CONVERGENCE PROTOCOL

We begin this section by recalling some graph-theoretical notions (e.g., see [7] and [21] for details). In particular, an undirected graph $\mathcal{G}$ is defined by a set $\mathcal{V}_{\mathcal{G}} = \{1,\ldots,N\}$ of nodes and a set $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$ of edges. If $(i,j) \in \mathcal{E}_{\mathcal{G}}$, then the nodes i and j are neighbors and the neighboring relation is indicated with $i \sim j$. The number of agent i's neighbors is its degree and denoted as d_i. The degree matrix of a graph $\mathcal{G}$, $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N\times N}$, is then defined by $\mathcal{D}(\mathcal{G}) \triangleq \mathrm{diag}(d)$ with

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$d = [d_1, \ldots, d_N]^{\mathrm{T}}$. Furthermore, a path $i_0 i_1 \ldots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $\mathcal{G}$ is called connected when there exists a path between any pair of distinct nodes. The adjacency matrix of a graph $\mathcal{G}$, $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{N \times N}$, is also defined by $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $(i, j) \in \mathcal{E}_{\mathcal{G}}$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. Finally, the standard Laplacian matrix, $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{R}}_+^{N \times N}$, is defined by $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ with $\mathbf{span}\{\mathbf{1}_N\}$ being its nullspace.

A. The New Laplacian Matrix

Consider a multiagent system with N nodes communicating under a connected and undirected graph $\mathcal{G}$ with the standard adjacency matrix $\mathcal{A}(\mathcal{G})$. Let $w = [w_1, \ldots, w_N] \in \mathbb{R}^N$ be a vector with positive elements (i.e., $w_i \in \mathbb{R}_+$ for all $i = 1, \ldots, N$), which is the representative vector for the desired nullspace $\mathbf{span}\{w\}$. We define the new, altered degree matrix $\bar{\mathcal{D}}(\mathcal{G}, w)$ as a diagonal matrix such that

$$[\bar{\mathcal{D}}(\mathcal{G}, w)]_{ii} = \sum_{j=1}^{N} \frac{[\mathcal{A}(\mathcal{G})]_{ij} w_j}{w_i} = \sum_{i \sim j} \frac{w_j}{w_i}, \tag{1}$$

or equivalently,

$$\bar{\mathcal{D}}(\mathcal{G}, w) \triangleq \mathrm{diag}(\mathcal{A}(\mathcal{G})\, w)(\mathrm{diag}(w))^{-1} \in \mathbb{R}^{N \times N}. \tag{2}$$

For simplicity, we now write $\bar{\mathcal{D}}$ for the new degree matrix defined in (2) and $\mathcal{A}$ for the standard adjacency matrix in this section, unless stated otherwise. Next, we define the new Laplacian matrix with the desired, user-assigned nullspace $\mathbf{span}\{w\}$ as

$$\bar{\mathcal{L}}(\mathcal{G}, w) \triangleq \bar{\mathcal{D}} - \mathcal{A} = \mathrm{diag}(\mathcal{A}w)(\mathrm{diag}(w))^{-1} - \mathcal{A}. \tag{3}$$

Note that when $w = \mathbf{1}_N$, $\bar{\mathcal{L}}(\mathcal{G}, w) \equiv \mathcal{L}(\mathcal{G})$, where $\mathcal{L}(\mathcal{G})$ is the standard Laplacian matrix. For the standard Laplacian matrix, $w_i = w_j = \frac{w_j}{w_i} = 1$ for all $i, j = 1, \ldots, N$; thus, the degree of agent i is simply the number of its neighbors. For the case where $w \neq \mathbf{span}\{\mathbf{1}_N\}$, agent i requires w_j value either by default (i.e., preprogrammed) or through information exchange. In what follows, we investigate the properties of the new Laplacian matrix $\bar{\mathcal{L}}(\mathcal{G}, w)$.

We first show that $\bar{\mathcal{L}}(\mathcal{G}, w)$ is a positive semidefinite matrix. For this purpose, consider the quadratic form of the new Laplacian matrix

$$\begin{aligned} x^{\mathrm{T}} \bar{\mathcal{L}}(\mathcal{G}, w) x =& x^{\mathrm{T}} (\bar{\mathcal{D}} - \mathcal{A}) x \\ =& x^{\mathrm{T}} \big(\mathrm{diag}(\mathcal{A}w)(\mathrm{diag}(w))^{-1} - \mathcal{A} \big) x \\ =& \sum_{i=1}^{N} \left(\sum_{i \sim j} \frac{w_j}{w_i} x_i^2 \right) - \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} 2 x_i x_j \\ =& \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} \left(\frac{w_j}{w_i} x_i^2 + \frac{w_i}{w_j} x_j^2 \right) - \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} 2 x_i x_j \\ =& \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} \left(\sqrt{\frac{w_j}{w_i}} x_i - \sqrt{\frac{w_i}{w_j}} x_j \right)^2 \geq 0. \end{aligned} \tag{4}$$

Therefore, $\bar{\mathcal{L}}(\mathcal{G}, w)$ is a positive semi-definite matrix. This implies that $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$, where λ_i, $i = 1, \ldots, N$ are the eigenvalues of the new Laplacian matrix $\bar{\mathcal{L}}(\mathcal{G}, w)$. In this paper, the indices of eigenvalues of the new Laplacian matrix follow the above order, unless stated otherwise. In addition, the last equality in (4) can be rewritten as

$$x^{\mathrm{T}} \bar{\mathcal{L}}(\mathcal{G}, w) x = \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} \left(\sqrt{\frac{|w_j|}{|w_i|}} x_i - \sqrt{\frac{|w_i|}{|w_j|}} x_j \right)^2 \geq 0. \tag{5}$$

Hence, (4) holds as long as all elements in w are nonzero and shares the same sign. That is, the new Laplacian (3) is defined under any vector w with nonzero elements and share the same sign (The result for w with nonzero elements and arbitrary signs will be presented in a future research). We note also that $\mathbf{span}\{w\}$ is in the nullspace of $\bar{\mathcal{L}}(\mathcal{G}, w)$ as

$$\begin{aligned} \bar{\mathcal{L}}(\mathcal{G}, w) w =& \mathrm{diag}(\mathcal{A}w)(\mathrm{diag}(w))^{-1} w - \mathcal{A}w \\ =& \mathrm{diag}(\mathcal{A}w) \mathbf{1}_N - \mathcal{A}w = \mathcal{A}w - \mathcal{A}w = \mathbf{0}_N. \end{aligned} \tag{6}$$

In fact, $\mathbf{span}\{w\}$ is the nullspace of $\bar{\mathcal{L}}(\mathcal{G}, w)$, which can be shown by contradiction.

Next, we assume orientation of each edge is assigned arbitrarily and define the new incidence matrix $\bar{E}(\mathcal{G}) \in \mathbb{R}^{N \times m}$ with m being the number of edges in the graph $\mathcal{G}$ as

$$[\bar{E}(\mathcal{G})]_{ik} = \begin{cases} -\sqrt{\frac{w_j}{w_i}} & \text{if } v_i \text{ is the tail of the edge } e_k = (i, j) \\ \sqrt{\frac{w_j}{w_i}} & \text{if } v_i \text{ is the head of the edge } e_k = (i, j) \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where $i = 1, \ldots, N$ and $k = 1, \ldots, m$. Therefore, the last equality in (4) can be rewritten as

$$\begin{aligned} x^{\mathrm{T}} \bar{\mathcal{L}}(\mathcal{G}, w) x =& \sum_{(i,j) \in \mathcal{E}_{\mathcal{G}}} \left(\sqrt{\frac{w_j}{w_i}} x_i - \sqrt{\frac{w_i}{w_j}} x_j \right)^2 \\ =& \| \bar{E}(\mathcal{G})^{\mathrm{T}} x \|_2^2 \\ =& x^{\mathrm{T}} \bar{E}(\mathcal{G}) \bar{E}(\mathcal{G})^{\mathrm{T}} x \geq 0. \end{aligned} \tag{8}$$

As a result, the new Laplacian matrix can now be rigorously defined in a similar way as the standard Laplacian matrix

$$\bar{\mathcal{L}}(\mathcal{G}, w) = \bar{\mathcal{D}} - \mathcal{A} = \bar{E}(\mathcal{G}) \bar{E}(\mathcal{G})^{\mathrm{T}}. \tag{9}$$

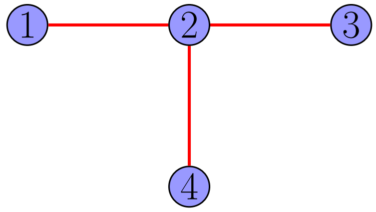
We are now ready to state the following theorem.

Theorem 1. *The graph $\mathcal{G}$ is connected if and only if $\lambda_2 > 0$.*

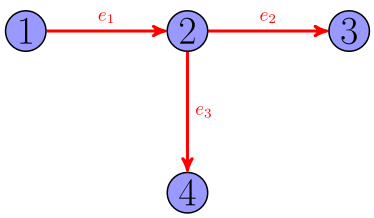
Due to page limitation, the proof of this result will be reported elsewhere. For interested readers, we note that $\bar{\mathcal{L}}(\mathcal{G}, w)$ and $\bar{E}(\mathcal{G})$ have the same nullspace. Hence, the proof follows in the same spirit of Theorem 2.8 of [7].

We now illustrate this result with an example. Consider an undirected and connected graph $\mathcal{G}$ with 4 nodes as shown in Figure 1(a). The adjacency matrix $\mathcal{A}$ of the corresponding graph $\mathcal{G}$ is

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{10}$$



(a)



(b)

Fig. 1. (a) An undirected and connected graph $\mathcal{G}$ with 4 nodes and (b) its oriented graph.

Let $w = [1, 2, 3, 4]^{\mathrm{T}}$. Then, the modified degree matrix is

$$\bar{\mathcal{D}} \triangleq \operatorname{diag}(\mathcal{A}w)(\operatorname{diag}(w))^{-1} = \operatorname{diag}([2, 4, \tfrac{2}{3}, \tfrac{1}{2}]). \tag{11}$$

As a result, we obtain the new Laplacian matrix in the form

$$\bar{\mathcal{L}}(\mathcal{G}, w) = \bar{\mathcal{D}} - \mathcal{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 \\ 0 & -1 & \frac{2}{3} & 0 \\ 0 & -1 & 0 & \frac{1}{2} \end{bmatrix}. \tag{12}$$

In addition, we can assign the orientation of the edge as in Figure 1(b) and construct the new incidence matrix $\bar{E}(\mathcal{G})$ based on (7) as

$$\bar{E}(\mathcal{G}) = \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ \sqrt{\frac{1}{2}} & -\sqrt{\frac{3}{2}} & -\sqrt{2} \\ 0 & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}} \end{bmatrix}. \tag{13}$$

It can be readily verified that $\bar{\mathcal{L}}(\mathcal{G}, w) = \bar{E}(\mathcal{G})\bar{E}(\mathcal{G})^{\mathrm{T}}$.

B. The Nullspace Convergence Algorithm

In this section, we introduce a new distributed algorithm for multiagent networked system to converge to a desired nullspace and discuss the stability of this algorithm. In particular, consider a multiagent system with N agents exchanging information according to a connected and undirected graph $\mathcal{G}$ and operating under the following distributed algorithm

$$\dot{x}_i(t) = -\sum_{i\sim j} \left(\frac{w_j}{w_i} x_i(t) - x_j(t) \right), \quad x_i(0) = x_{i0}, \tag{14}$$

or equivalently,

$$\dot{x}_i(t) = -\sum_{i\sim j} (x_i(t) - x_j(t)) + \sum_{i\sim j} \left(1 - \frac{w_j}{w_i}\right) x_i(t),$$

$$x_i(0) = x_{i0}. \tag{15}$$

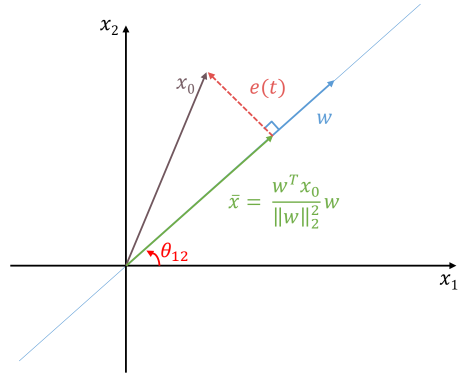


Fig. 2. The initial vector x_0 is projected onto the nullspace represented by vector w.

Above, $x_i(t) \in \mathbb{R}$ stands for the state of agent i, $i = 1, \ldots, N$, and $w_i \in \mathbb{R}_+$ stands for the element i-th of the desired nullspace $w \triangleq [w_1, \ldots, w_N] \in \mathbb{R}^N$.

For stability discussion purposes, let $x(t) \triangleq [x_1, \ldots, x_N]^{\mathrm{T}} \in \mathbb{R}^N$ be the aggregated vector. Then, one can compactly write (15) as

$$\dot{x}(t) = -\bar{\mathcal{L}}(\mathcal{G}, w)x(t), \quad x(0) = x_0, \tag{16}$$

with $\bar{\mathcal{L}}(\mathcal{G}, w)$ being the new Laplacian matrix defined in (3).

Theorem 2. *Consider an undirected and connected graph $\mathcal{G}$ with N nodes and the new Laplacian matrix $\bar{\mathcal{L}}(\mathcal{G}, w)$ defined by (3), where w is a vector with positive elements. Under the distributed protocol given by (15) or the compact form given by (16), $x(t)$ exponentially converges to $\frac{(w^{\mathrm{T}}x_0)}{\|w\|_2^2} w$.*

The proof of this theorem can be shown by investigating the explicit solution of (16) and utilizing the fact that $\bar{\mathcal{L}}(\mathcal{G}, w)$ is diagonalizable. Note that $\frac{(w^{\mathrm{T}}x_0)}{\|w\|_2^2} w \equiv \frac{w^{\mathrm{T}}x_0}{\|w\|_2} \frac{w}{\|w\|_2}$ is a vector projection, which indicates that under the protocol (16), the initial vector x_0 is projected onto the nullspace of $\bar{\mathcal{L}}(\mathcal{G}, w)$ (i.e., **span**$\{w\}$). Figure 2 simply illustrates this viewpoint for a network with two agents. In addition, the ratio between two agents can be viewed in term of angles. Let θ_{ij} be the angle between x_i-axis and the desired nullspace, then $\theta_{ji} = \pi/2 - \theta_{ij}$ and $\tan(\theta_{ij}) = \frac{w_j}{w_i} = 1/\tan(\theta_{ji})$. This fact shows that the standard Laplacian matrix is a special case where $\theta_{ij} = \theta_{ji} = \pi/4$ that leads to the ratio $\frac{w_i}{w_j} = \frac{w_j}{w_i} = 1$. Furthermore, define the error $e(t) \triangleq x(t) - \frac{(w^{\mathrm{T}}x_0)}{\|w\|_2^2} w$. Once again, by taking the advantage of the explicit solution of (16), it can be shown that $w^{\mathrm{T}}e(t) = 0$, that is, $e(t)$ is orthogonal to the nullspace represented by w.

The proposed distributed protocol (15) generalizes a wide range of nullspace convergence including the so-called average consensus protocol. Specifically, when $w = \textbf{span}\{\mathbf{1}_N\}$, then $w_i = w_j$ for all i, $j = 1, \ldots, N$ and the second term of (15) is eliminated yielding the structure of the average consensus protocol. In addition, the second term of (15) shows that the nullspace can be changed by adding self-loops (i.e., modifying the degree of agent i) to the standard average consensus protocol. While the approach in [19] and [20] requires the exact knowledge of each neighboring agent's state (i.e., $x_j(t)$), the protocol (15) only requires each agent to know the "distance" to its neighbor(s) (i.e., $(x_i(t) - x_j(t))$. Therefore, our approach has the potential to require less information exchange compare to the method in [19] and [20]

III. Nullspace Control with the Leader-Follower Algorithm

A. Convergence to a Specific Vector in the Nullspace

In this section, we extend the result of Section II to its leader-follower version and show that this new algorithm can be directly applied to drive the multiagent system toward a specific vector in the nullspace of the new Laplacian matrix. In particular, this new leader-follower algorithm is different from the standard leader-follower algorithm in the sense that while the leader tracks the command, the followers arrange themselves relative to their neighbors such that the ratios $\frac{x_i(t)}{x_j(t)} = \frac{w_i}{w_j}$ for all $(i,j) \in \mathcal{E}_\mathcal{G}$ are satisfied. Specifically, consider a multiagent system with N agents exchanging information according to a connected and undirected graph $\mathcal{G}$ and operating under the distributed algorithm in the form given by

$$\dot{x}_i(t) = -\sum_{i\sim j}(x_i(t) - x_j(t)) + \sum_{i\sim j}\left(1 - \frac{w_j}{w_i}\right)x_i(t) - k_i\left(x_i(t) - c(t)\right), \quad x_i(0) = x_{i0}. \tag{17}$$

In (17), $x_i(t) \in \mathbb{R}$ stands for the state of agent i, $i = 1,\ldots,N$, $w_i \in \mathbb{R}_+$ stands for the element i-th of the desired nullspace $w \triangleq [w_1,\ldots,w_N] \in \mathbb{R}^N$, and $c(t) \in \mathbb{R}$ stands for the time-varying tracking command with bounded time rate of change (i.e., $|\dot{c}(t)| \leq \bar{c}$ with $\bar{c} \in \mathbb{R}_+$). In addition, we assume that the network has at least one leader ($k_i = 1$ when agent i is a leader and $k_i = 0$ otherwise). Here, the tracking command $c(t)$ is only available to the leader(s).

Let $x(t) \triangleq [x_1,\ldots,x_N]^{\mathrm{T}} \in \mathbb{R}^N$ be the aggregated vector. One can then rewrite (17) as

$$\dot{x}(t) = -Fx(t) + K\mathbf{1}_N c(t), \quad x(0) = x_0. \tag{18}$$

In (18), $F \triangleq \bar{\mathcal{L}}(\mathcal{G},w) + K \in \mathbb{R}^{N\times N}$ with $K = \mathrm{diag}([k_1,\ldots,k_N]) \in \mathbb{R}^{N\times N}$ and $\bar{\mathcal{L}}(\mathcal{G},w)$ is the new Laplacian matrix defined in (3). We are now ready for the following theorem.

Theorem 3. *Consider a connected and undirected graph $\mathcal{G}$ with N nodes and the new Laplacian matrix $\bar{\mathcal{L}}(\mathcal{G},w)$ defined by (3), where w is a vector with positive elements. Under the distributed protocol given by (17) or its compact form given by (18), $x(t)$ approaches to the neighborhood of $F^{-1}K\mathbf{1}_N c(t)$ as $t \to \infty$.*

The proof of this theorem is omitted due to page limitation and will be reported elsewhere. For interested readers, it follows by first showing that F is positive definite, then defining $e(t) \triangleq x(t) - F^{-1}K\mathbf{1}_N c(t)$ and taking its time derivative to obtain the error dynamics. Finally, Lyapunov analysis can be utilized to acquire the result. Note that when the tracking command $c(t)$ is constant, the closed-loop error $e(t)$ exponentially goes to 0.

Since $Fw = \bar{\mathcal{L}}(\mathcal{G},w)w + Kw = Kw$, we have $w = F^{-1}Kw$. In addition, without loss of generality, the tracking command can be written as $c(t) = \gamma(t)w_i$, where w_i is the leader's corresponding component in the desired nullspace vector w, then by definition of K matrix, we have $K\mathbf{1}_N c(t) = \gamma(t)Kw$. As a result, $F^{-1}K\mathbf{1}_N c(t) = \gamma(t)F^{-1}Kw = \gamma(t)w$. This indicates that under the protocol (18), $x(t)$ converges to the neighborhood of $\gamma(t)w$.

B. An Application to Formation Control

In this section, we utilize the result of Section III-A and the multiplex information network architecture proposed in [22] to allow formation scaling and tracking in a distributed manner. First, we note that multiplex information network architecture describes a multiagent system with multiple layers of information exchange including intralayer and interlayer communication links. For the 2D formation tracking problem, we use the standard formation translation algorithm as the main layer and the algorithm (17) as the second layer to update the desired relative position of each agent in the formation. Mathematically speaking, consider a group of N vehicles communicating with each other under a connected and undirected graph and operating under the following algorithm

$$\dot{x}_i(t) = -\sum_{i\sim j}\left((x_i(t) - \xi_i(t)) - (x_j(t) - \xi_j(t))\right) - k_i(x_i(t) - \xi_i(t) - c_x(t)), \quad x_i(0) = x_{i0}, \tag{19}$$

$$\dot{\xi}_i(t) = -\sum_{i\sim j}(\xi_i(t) - \xi_j(t)) + \sum_{i\sim j}\left(1 - \frac{w_{xj}}{w_{xi}}\right)x_i(t) - k_i\left(\xi_i(t) - \gamma w_i\right), \quad \xi_i(0) = \xi_{i0}, \tag{20}$$

where $x_i(t)$ and $\xi_i(t) \in \mathbb{R}$, $i = 1,\ldots,N$ are the current position and the desired relative position of agent i in x-axis, respectively[1]; $w_x \triangleq [w_{x1},\ldots,w_{xN}] \in \mathbb{R}^N$ is a constant vector which represents the desired baseline formation in x-axis of the agent teams with $w_{xi} \in \mathbb{R}_+$ denoting the desired relative position of individual agent i; and $c_x(t) \in \mathbb{R}$ is a time-varying tracking command for the formation in x-axis with bounded time rate of change (i.e., $|\dot{c}_x(t)| \leq \bar{c}_x$ where $\bar{c}_x \in \mathbb{R}_+$). Along the lines of the discussion in the last paragraph of Section III-A, the tracking command for the second layer (20) is now explicitly in the form of γw_{xi}, where $\gamma \in \mathbb{R}_+$ plays the role as the command scaling factor for the formation. In addition, we consider the network has at least one leader, where $k_i = 1$ if agent i is the leader and $k_i = 0$ otherwise. Thus, the tracking command $c_x(t)$ and the scaling factor γ are only available to the leader(s).

For stability analysis, we define the position error as

$$\tilde{x}_i(t) \triangleq x_i(t) - \xi_i(t) - c_x(t), \tag{21}$$

and taking its time derivative to obtain

$$\dot{\tilde{x}}_i(t) = -\sum_{i\sim j}(\tilde{x}_i(t) - \tilde{x}_i(t)) - k_i\tilde{x}_i(t) - \dot{\xi}_i(t) - \dot{c}_x(t), \quad \tilde{x}_i(0) = \tilde{x}_{i0}. \tag{22}$$

[1]The same structure is utilized for y-axis, and hence, omitted.

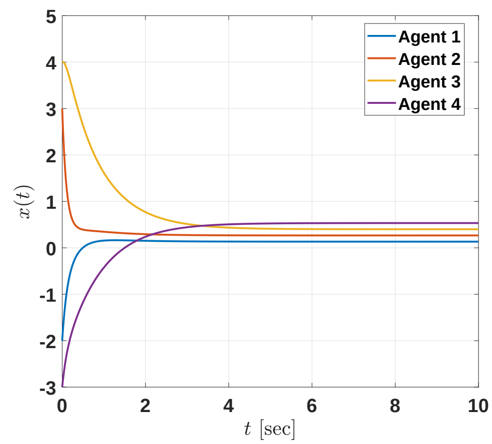


Fig. 3. The evolution of $x(t)$ under the protocol (15) in Example 1.

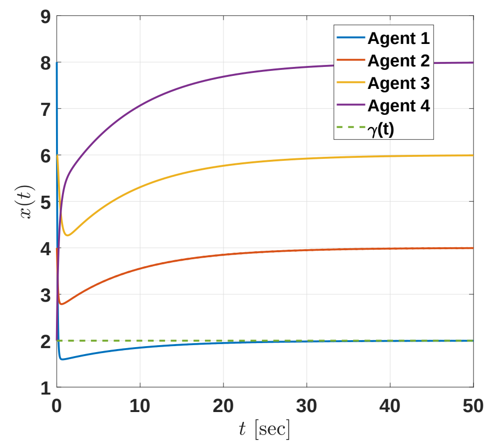


Fig. 4. The evolution of $x(t)$ under the protocol (18) in Example 2 with $\gamma(t) = 2$.

Let $\tilde{x}(t) \triangleq [\tilde{x}_1, \ldots, \tilde{x}_N]^{\mathrm{T}} \in \mathbb{R}^N$ and $\xi \triangleq [\xi_1, \ldots, \xi_N]^{\mathrm{T}} \in \mathbb{R}^N$ be the aggregated vectors. Then, (22) can be written in the compact form given by

$$\dot{\tilde{x}}(t) = -G\tilde{x}(t) - \dot{\xi}(t) - \dot{c}_x(t)\mathbf{1}_N, \quad \tilde{x}(0) = \tilde{x}_0 \tag{23}$$

where $G \triangleq \mathcal{L}(\mathcal{G}) + K$ with $K = \mathrm{diag}([k_1, \ldots, k_N]) \in \mathbb{R}^{N\times N}$ and $\mathcal{L}(\mathcal{G})$ is the standard Laplacian matrix. We note that G is a positive definite matrix (e.g., see Lemma 3.3 of [8]).

We now ready state the following theorem.

Theorem 4. *Consider the networked multiagent system given by (19) and (20), where agents exchange their local measurements under a connected and undirected graph $\mathcal{G}$. Then, the closed-loop error dynamics given by (23) is input-to-state stable with the term $-\dot{\xi}(t) - \dot{c}_x(t)\mathbf{1}_N$ being considered as the input.*

Utilizing the facts that $\dot{\xi}(t)$ is bounded by Theorem 3, $\dot{c}_x(t)\mathbf{1}_N$ is bounded by assumption, and the matrix G is positive definite, the result is immediate. Note that when $\tilde{x}(t)$ approaches 0, $\tilde{x}_i(t) = \tilde{x}_j(t) = 0$. As a result, $x_i(t) - \xi_i(t) - c_x(t) = x_j(t) - \xi_j(t) - c_x(t)$, or equivalently, $x_i(t) - \gamma w_{xi} - c_x(t) = x_j(t) - \gamma w_{xj} - c_x(t)$ for all $i, j = 1, \ldots, N$, which indicates that agents achieve the formation and are tracking the command.

IV. Illustrative Numerical Examples

In this section, we consider several examples to illustrate four contribution. Specifically, we consider a group of 4 agents communicating under a connected and undirected graph $\mathcal{G}$ as depicted in Figure 1(a) for all following examples.

Example 1. This example aims to illustrate the protocol given by(15). Initially, the state of agents are set to $x(0) = [-2; 3; 4; -3]$. The desired nullspace is chosen to be the span of the representative vector $w = [1; 2; 3; 4]$ and the protocol given by (15) is utilized. To accelerate the convergence, a gain $a = 2$ is used to multiply the protocol. Figure 3 shows that agents converge to vector $\frac{(w^{\mathrm{T}}x_0)}{\|w\|_2^2}w = [0.1333; 0.2667; 0.4000; 0.5333]$ as expected from Theorem 2.

Example 2. This example aims to illustrate the result of Theorem 3. For this example, agent 1 is chosen as the leader. The initial condition is set to $x(0) = [8, 4, 6, 2]^{\mathrm{T}}$ and the desired nullspace is chosen as $w = [1, 2, 3, 4]^{\mathrm{T}}$. In this case, the protocol (17) is implemented and as discussed in the last paragraph of Section III-A, we should choose the command $c(t) = \gamma(t)w_1 \equiv \gamma(t)$. Once again, the protocol is multiplied by a gain $a = 5$ to accelerate the convergence. Figure 4 shows that all agents converges to the $2w$ when $\gamma(t) = 2$.

Example 3. In this example, we illustrate the result of Theorem 4. Specifically, we choose agent 3 to be the leader in this case. Initially, agents are located at $(x_i, y_i) = (-6, 2), (-5, 4), (-5, 7), (-2, -1)$ for $i = 1, \ldots, 4$. The initial desired relative positions of agents in the formation (ξ_{xi}, ξ_{yi}) are set to $(6, 6), (2, 2), (0, 3), (1, 1)$ for $i = 1, \ldots, 4$, while the actual desired formation (diamond shape) is encoded within $(w_{xi}, w_{yi}) = (2, 3), (3, 2), (2, 1), (1, 2)$ for $i = 1, \ldots, 4$. The tracking command is set to $(c_x(t), c_y(t)) = (0.1t, 2.5\sin(0.05t))$. As discussed in Section III-B, γ plays the role as the scaling factor. Therefore, to see its effect, for the first 50 seconds, we set $\gamma = 1$ and for the last 50 seconds, we set $\gamma = 2$. Under the proposed protocol given by (19) and (20), agents achieved the desired the formation while tracking the command as illustrated in Figure 5. In addition, at t = 50 seconds, the formation size is doubled as expected.

V. Conclusion

We generalized the standard Laplacian matrix, which has a nullspace spanning the vector of ones, through introducing a new Laplacian matrix, which has a user-assigned nullspace spanning any vector with positive elements. Focusing on undirected and connected graphs, the mathematical definition of this Laplacian matrix was given and its fundamental properties were shown. Distributed control architecture were then presented for convergence to the desired nullspace and for convergence to a specific vector within that nullspace. An application of the proposed Laplacian matrix to formation tracking and scaling problem was also given, and several illustrative numerical examples were shown to complement our theoretical results. We believe that the contribution of this

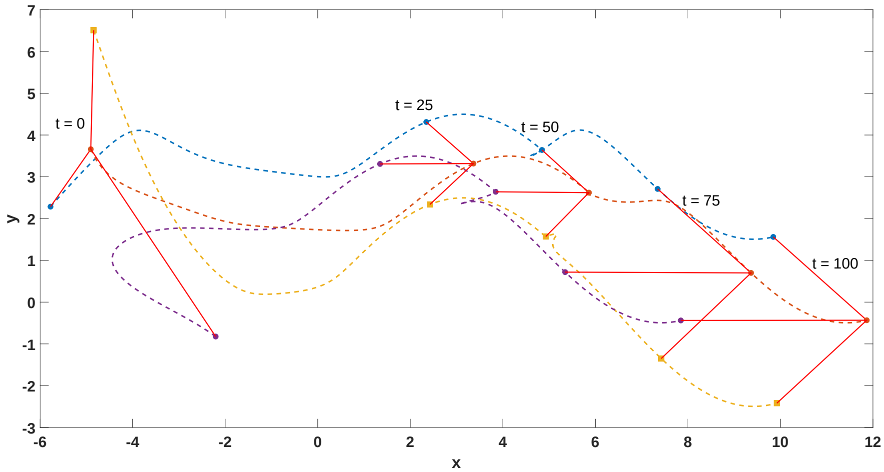


Fig. 5. Response of the multiagent system under the proposed control architecture given by (19) and (20) in Example 3. Circles and square denote agents' position at some specific time instants where the square denotes the leader, dashed lines denote agents' trajectories, and solid lines denote the communication links between agents.

paper will open up many research directions to investigate from here toward composing complex cooperative behaviors in multiagent systems through nullspace assignment and control.

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References

[1] R. M. Murray, K. J. Astrom, S. P. Boyd, R. W. Brockett, and G. Stein, "Future directions in control in an information-rich world," *IEEE Control Systems Magazine*, vol. 23, no. 2, pp. 20–33, 2003.

[2] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *American Control Conference*, 2005, pp. 1859–1864.

[3] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control systems magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[4] F. Lamnabhi-Lagarrigue, A. Annaswamy, S. Engell, A. Isaksson, P. Khargonekar, R. M. Murray, H. Nijmeijer, T. Samad, D. Tilbury, and P. Van den Hof, "Systems & control for the future of humanity, research agenda: Current and future roles, impact and grand challenges," *Annual Reviews in Control*, vol. 43, pp. 1–64, 2017.

[5] X. Ge, Q.-L. Han, D. Ding, X.-M. Zhang, and B. Ning, "A survey on recent advances in distributed sampled-data cooperative control of multi-agent systems," *Neurocomputing*, vol. 275, pp. 1684–1701, 2018.

[6] W. Ren and R. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*. Springer, 2007.

[7] M. Mesbahi and M. Egerstedt, *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.

[8] F. L. Lewis, H. Zhang, K. Hengster-Movric, and A. Das, *Cooperative control of multi-agent systems: Optimal and adaptive design approaches*. Springer Science & Business Media, 2014.

[9] M. Mesbahi and F. Y. Hadaegh, "Formation flying control of multiple spacecraft via graphs, matrix inequalities, and switching," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 2, pp. 369–377, 2001.

[10] A. K. Das, R. Fierro, V. Kumar, J. P. Ostrowski, J. Spletzer, and C. J. Taylor, "A vision-based formation control framework," *IEEE transactions on robotics and automation*, vol. 18, no. 5, pp. 813–825, 2002.

[11] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on automatic control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[12] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Flocking in fixed and switching networks," *IEEE Transactions on Automatic control*, vol. 52, no. 5, pp. 863–868, 2007.

[13] W. Ren and N. Sorensen, "Distributed coordination architecture for multi-robot formation control," *Robotics and Autonomous Systems*, vol. 56, no. 4, pp. 324–333, 2008.

[14] F. Xiao and L. Wang, "Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays," *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1804–1816, 2008.

[15] M. M. Zavlanos, H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Hybrid control for connectivity preserving flocking," *IEEE Transactions on Automatic Control*, vol. 54, no. 12, pp. 2869–2875, 2009.

[16] Z. Li, G. Wen, Z. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs," *IEEE Transactions on Automatic Control*, vol. 60, no. 4, pp. 1152–1157, 2015.

[17] S. Su and Z. Lin, "Distributed consensus control of multi-agent systems with higher order agent dynamics and dynamically changing directed interaction topologies," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 515–519, 2016.

[18] X.-K. Liu, Y.-W. Wang, J.-W. Xiao, and W. Yang, "Distributed hierarchical control design of coupled heterogeneous linear systems under switching networks," *International Journal of Robust and Nonlinear Control*, vol. 27, no. 8, pp. 1242–1259, 2017.

[19] L. D. DeVries and M. D. Kutzer, "Kernel design for coordination of autonomous, time-varying multi-agent configurations," in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 1975–1980.

[20] L. DeVries, A. Sims, and M. D. Kutzer, "Kernel design and distributed, self-triggered control for coordination of autonomous multi-agent configurations," *Robotica*, vol. 36, no. 7, pp. 1077–1097, 2018.

[21] C. D. Godsil, G. Royle, and C. Godsil, *Algebraic graph theory*. Springer New York, 2001, vol. 207.

[22] D. Tran, T. Yucelen, and E. L. Pasiliao, "Formation control with multiplex information networks," *IEEE Transactions on Control Systems Technology*, 2018.